

A NOTE ON THE ANALYTIC APPROXIMATIONS OF THE H-FUNCTION

by

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Received on July 15, 2004; accepted on September 17, 2004

This paper shows some analytic approximations of the H-function derived earlier and obtained by decomposition of the angular flux density of particles combined with the zero order DPN method. A novel solution based on the modified DPN method has been presented too. The comparison with well-known Chandrasekhar's results obtained by numerical procedure and with Pomraning's analytic approximations has been performed.

Key words: H-function, DPN method, reflection coefficient

INTRODUCTION

Since the early days of studying of the transport equation, the fundamental H-function has been introduced as a part of solution to the problem of particle reflection from the homogenous target. The values of the function have been numerically determined with great precision and tabled in detail [1], and for a long time its thorough mathematical studying and application has been one of the central questions of the transport theory, especially of the particle albedo problem.

Beside numerical determination of the H-function by direct iteration of the characteristic non-linear equation, analytic iterative solutions of the same equation have been attempted as well. One of the first strivings started from the results obtained by Case's singular eigenfunctions method, and conveniently chosen test function [2]. Even though the H-function was determined in the second iteration with precision to the fourth important digit, the procedure showed as very clumsy and hermetic because the final result included complex expressions containing new fundamental functions

(for example, the X-function, well known in Case's theory). The problem of the particle transport in half-space and the related task of determining the H-function have been solved lately by multiple collision method [3]; however, this procedure has been judged as complicated and impractical [4].

Not long ago, it has been noted that the procedures of the analytic approximation of the H-function based on a verified approximation method were not used enough. Namely, as already said, the advantage was given to numerical integration of the characteristic non-linear integral equation of this function and its very precise calculation. However, the greatest potential advantage of analytic approximations has been the possibility to solve the more complex energy albedo problem as well in a consistent analytic manner, which can be presented by the H-function. Therefore, a few analytic approximations of the H-function have been recently proposed [5, 6] based on the standard and modified DPN method [7, 8]. Another form of the analytic approximation of the H-function has been presented in this paper, which is especially efficient with the lower values of the particle multiplication parameter c , and is based on the specific decomposition of the angular flux density and on the zero order DPN method.

ANALYTIC APPROXIMATIONS OF THE H-FUNCTION

In the case of monoenergetic particle transport in plane geometry, the classical transport the-

Scientific paper
UDC: 539.125/.126
BIBLID: 1451-3994, 19 (2004), 2, pp. 47-51

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ory has given an exact solution for the angular flux density of the reflected particles $\phi(0, \mu, \mu_0)$ [1, 4]

$$\phi(0, \mu, \mu_0) = \frac{c}{2} \frac{H(c, \mu)H(c, \mu_0)}{\mu_0 \mu}, \mu > 0 \quad (1)$$

and for the half-space reflection coefficient $\gamma(c, \mu_0)$

$$\gamma(c, \mu_0) = \frac{\int_0^1 \mu \phi(0, \mu, \mu_0) d\mu}{\int_0^1 \frac{\mu}{\mu_0} \delta(\mu - \mu_0) d\mu} = 1 - \sqrt{1 - c} H(c, \mu_0) \quad (2)$$

where the H-function $H(c, \mu)$ is defined as a solution of the nonlinear integral equation

$$\frac{1}{H(c, \mu)} = \sqrt{1 - c} \int_0^1 \frac{d\mu'}{\mu'} \frac{\mu' H(c, \mu')}{\mu - \mu'} \quad (3)$$

In eqs. (1-3) μ stands for the cosine of the angle between the vector of particle velocity and the normal on boundary surface at $x = 0$ oriented inside of the medium; μ_0 is the cosine of the angle of incidence of the plane source of initial particles which cross the boundary surface, $S(0, \mu, \mu_0) = \delta(\mu - \mu_0) / \mu_0, \mu_0 > 0$, while c presents the particle multiplication parameter. It is also assumed that the particle scattering function has an isotropic form.

If one compares the exact solution for the half-space reflection coefficient $\gamma(c, \mu_0)$, given by eq. (2), with the approximate formulae for $\gamma(c, \mu_0)$ derived previously through the ordinary and modified DPN procedures [9], then the approximate analytic expressions for the H-function can be obtained almost directly.

The ordinary DPN method

Starting with the flux decomposition

$$\phi(x, \mu, \mu_0) = \phi^{(0)}(x, \mu, \mu_0) + \phi^s(x, \mu, \mu_0) \quad (4)$$

where $\phi^{(0)}$ and ϕ^s denote the angular density fluxes of unscattered and scattered particles at the distance x inside the medium, respectively, and applying the ordinary DP0 approximation [7], one gets

$$\gamma_{DP0}(c, \mu_0) = \frac{1 - \sqrt{1 - c}}{1 - 2\mu_0 \sqrt{1 - c}} \quad (5)$$

and

$$H_{DP0}(c, \mu_0) = \frac{1 - 2\mu_0}{1 - 2\sqrt{1 - c} \mu_0} \quad (6)$$

The first modified DPN procedure

Based on the flux splitting

$$\begin{aligned} \phi(x, \mu, \mu_0) &= \phi^{(0)}(x, \mu, \mu_0) \\ \phi^{(1)}(x, \mu, \mu_0) &= u(\mu) \tilde{\phi}(x, \mu, \mu_0) \end{aligned} \quad (7)$$

with $\phi^{(1)}$ representing the angular flux density of the single scattered particles, $u(\mu)$ – Kronecker's unit step-function and $\tilde{\phi}$ – the angular flux density of the rest of particles, and the ordinary DP0 approximation, it is derived [6, 8]

$$\begin{aligned} \gamma_{M1}(c, \mu_0) &= \frac{c}{2} \left(1 - \mu_0 \ln \frac{1}{\mu_0} \right) \\ &= \frac{1 - \sqrt{1 - c}}{2} \frac{1 - \sqrt{1 - c} - c\mu_0 \ln \frac{1}{\mu_0}}{1 - 2\mu_0 \sqrt{1 - c}} \end{aligned} \quad (8)$$

and

$$\begin{aligned} H_{M1}(c, \mu_0) &= \frac{1 - 2\mu_0}{1 - 2\sqrt{1 - c} \mu_0} \\ &= \frac{c}{2} \frac{(1 - 2\mu_0)\mu_0 \ln(1 - 1/\mu_0) - 2\mu_0}{1 - 2\sqrt{1 - c} \mu_0} \end{aligned} \quad (9)$$

The second modified DPN procedure – new approximation of the H-function

Applying the angular flux decomposition

$$\begin{aligned} \phi(x, \mu, \mu_0) &= \phi^{(0)}(x, \mu, \mu_0) \\ \phi^a(x, \mu, \mu_0) &= u(\mu) \tilde{\phi}(x, \mu, \mu_0) \end{aligned} \quad (10)$$

in which ϕ^a represents the particles scattered after each collision strictly into directions oriented toward the boundary plane $x = 0$ [10],

$$\phi^a(x, \mu, \mu_0) = \frac{c}{2} \frac{1}{\mu_0 \mu} \frac{e^{-x/\mu_0}}{\tilde{\chi}(c, \mu_0)}, \mu > 0 \quad (11)$$

and the modified DP0 procedure, the reflection coefficient $\gamma_{M2}(c, \mu_0)$ is obtained

$$\begin{aligned} \gamma_{M2}(c, \mu_0) &= \frac{c}{2} \frac{1 - \mu_0 \ln \frac{1}{\mu_0}}{\tilde{\chi}(c, \mu_0)} \\ &= \frac{1}{2} \frac{(1 - \sqrt{1 - c})^2}{(1 - 2\mu_0 \sqrt{1 - c}) \tilde{\chi}(c, \mu_0)} \end{aligned} \quad (12)$$

The comparison of the exact expression (2) with the approximate result (12) brings to the third approximation for the H-function:

$$H_{M_2}(c, \mu) \approx \frac{1}{(1 - 2\sqrt{1 - c\mu_0})} \frac{(2 - c)\mu_0}{\tilde{\chi}(c, \mu_0)} \quad (13)$$

In eqs. (11-13) the new function $\tilde{\chi}(c, \mu)$ is introduced, defined by [10]

$$\tilde{\chi}(c, \mu) = 1 + \frac{c\mu}{2} \ln \left(1 + \frac{1}{\mu} \right) \quad (14)$$

In the previous expressions, the origin of approximations of the half-space reflection coefficient γ and the H-function is emphasized by the subscripts DP0, M_1 , and M_2 .

COMPARISON WITH REFERENT CHANDRASEKHAR'S RESULTS AND POMRANING'S APPROXIMATIONS

Formulae obtained by modification of DP0 method show remarkable accuracy, especially for small values of the parameter c . Table 1 gives the comparison of values of the approximation $H_{M_2}(c, \mu)$ with Chandrasekhar's calculations (marked with H_{CH}), exact to the sixth decimal digit. For $c = 0.1$, the maximal result deviation is to three units of the sixth digit, for $c = 0.3$, the difference is to three units of the fifth digit, and for $c = 0.6$, the difference is to the two units of the fourth digit.

However, we compare the accuracy of the analytic formulae (6), (9), and (13) and the two Pomraning's approximations of the H-function derived recently [11]:

$$H_{P_1}(c, \mu) \approx \frac{cJ}{(v - 1)} \gamma_+ \gamma_- \frac{v - \mu}{v \mu} \quad (15)$$

Table 1. Values of the $H_{M_2}(c, \mu)$ approximation and Chandrasekhar's H-function for $\mu \in [0.1]$ and three values of the parameter c

μ	$H(c, \mu)$					
	$c = 0.1$		$c = 0.3$		$c = 0.6$	
	H_{M_2}	H_{CH}	H_{M_2}	H_{CH}	H_{M_2}	H_{CH}
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	1.01236	1.01238	1.03968	1.03989	1.09043	1.09137
0.2	1.01863	1.01864	1.06103	1.06115	1.14462	1.14517
0.3	1.02300	1.02300	1.07635	1.07637	1.18588	1.18587
0.4	1.02631	1.02630	1.08819	1.08811	1.21916	1.21861
0.5	1.02894	1.02892	1.09771	1.09756	1.24685	1.24581
0.6	1.03108	1.03106	1.10558	1.10538	1.27036	1.26893
0.7	1.03287	1.03284	1.11222	1.11198	1.29064	1.28888
0.8	1.03439	1.03436	1.11791	1.11763	1.30835	1.30631
0.9	1.03570	1.03567	1.12285	1.12254	1.32396	1.32171
1.0	1.03684	1.03682	1.12717	1.12685	1.33784	1.33541

and

$$H_{P_2}(c, \mu) \approx \frac{c^2 J}{2(v - 1)} \gamma_+ \gamma_- \frac{v - \mu}{v \mu} \mu \ln \frac{1 - \mu}{\mu} + \frac{c^2 v J}{2(v - 1)} \gamma_+ \ln \frac{v}{v - 1} + \gamma_- \frac{v - \mu}{v \mu} \ln \frac{v - 1}{v} \quad (16)$$

Equations (15) and (16) are obtained in the course of the studying of the asymptotic flux behavior in half-space using the variational method and the singular eigenfunctions method. The constants J , γ_+ , and γ_- are defined by

$$J = \frac{(v - 1)e^{z_0/v}}{c\sqrt{\gamma}} \quad (17)$$

$$\gamma_{\pm} = \frac{1}{2} \frac{cv^2}{v^2 - 1} \pm 1 \quad (18)$$

and

$$\gamma_{\pm} = 2\gamma_{\pm} \frac{\ln \frac{v - 1}{v} \pm \frac{1}{v - 1}}{\ln \frac{v^2}{v^2 - 1}} \quad (19)$$

while the eigenvalue v is the positive root of the characteristic eq. [12]

$$\frac{cv}{2} \ln \frac{v - 1}{v} \pm 1 = 1 \quad (20)$$

It is obvious from eq. (17) that the parameter J is related to the extrapolated endpoint z_0 of the Milne problem, which accurate values are previously numerically determined [13].

Figure 1(a)–(c) compares the relative error ϵ_H of the formulae H_{DP0} , H_{M_1} and H_{M_2} calculated by eqs. (6), (9) and (13) with the relative error of the Pomraning's expressions H_{P_1} and H_{P_2} calculated by eqs. (15) to (19) for three values of the parameter c ($c = 0.3$, $c = 0.6$, and $c = 0.9$). For each value of the parameter c the worst analytic approximation H_{DP0} (6) is presented as well as the one of two approximations obtained by the modified DP0 method – expressions (9) or (13), which is more accurate for the selected value of the parameter c .

The two Pomraning's formulae H_{P_1} and H_{P_2} do not approach the exact value of the H-function for $\mu = 0$, but they are very accurate for $\mu = 1.0$. However, for any value of the parameter $c = 0.9$ it is possible to find at least one of the two approximations (9) or (13) which is more or equally accurate as the best Pomraning's approximation. Moreover, the approximations H_{P_1} and H_{P_2} have some other weaknesses which were common for the previous analytical approximations of the H-function [2, 3]. Namely, these expressions are rather

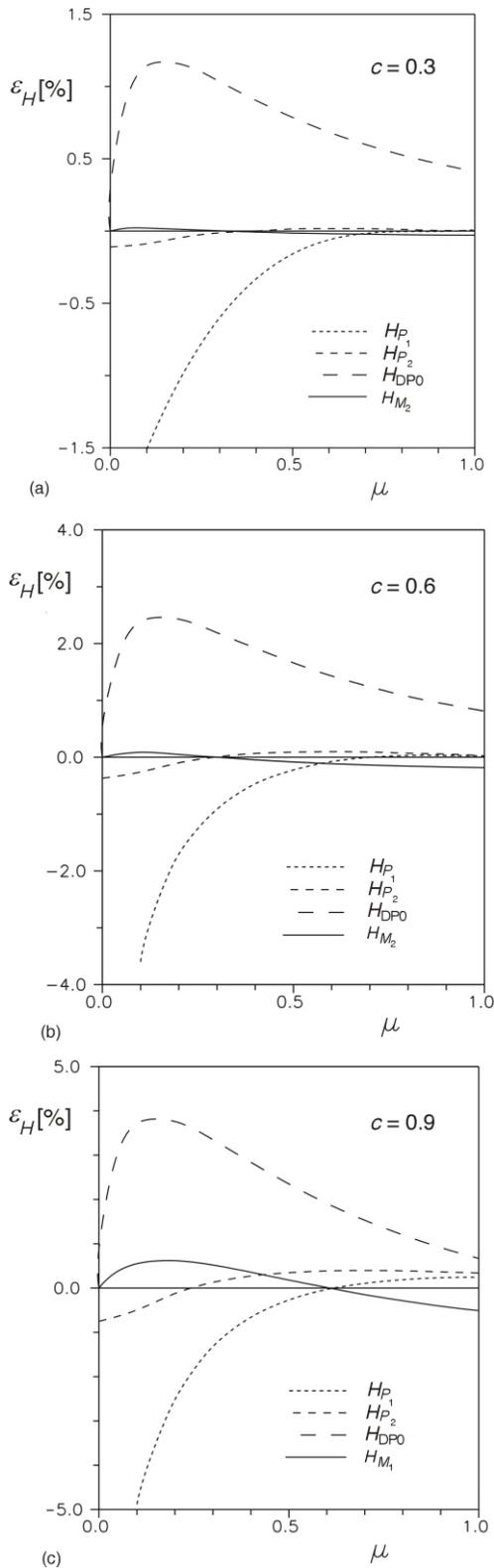


Figure 1. Comparison of the H-function analytic approximations with Pomraning's expressions for three values of the multiplication parameter c

complicated and very unusable for the consistent analytical procedures. They also include the new parameters specific for the different external transport methods on

which the proposed solution depend (for example, the eigenvalue ν of the singular eigenfunctions method and the extrapolated endpoint z_0 of the variational method).

Further improvement in the precision of the analytic approximations of the H-function, if at all possible in form of a simple expression, should be looked for in the ranges of the values $c < 1$ and $\mu < 1$.

CONCLUSIONS

Analytic formulae (6), (9) and (13) are exact for $\mu = 0$ and $c = 0$, which is not often the characteristic of other analytic approximations. They are of a simple shape and do not include parameters taken from another transport methods. Two approximations, $H_{M_1}(c, \mu)$ and $H_{M_2}(c, \mu)$, made by modification of DP0 procedure, are of high precision. This applies especially to the formula $H_{M_2}(c, \mu)$ (13) when $c = 0.6$, which is by accuracy close to the numerical approximations. Being very simple, formulae are easily applicable to the calculation of the energy dependent transport problem in which the Chandrasekhar's H-function appears as a component of the final solution.

ACKNOWLEDGEMENT

The paper contains the results achieved within the research financed by the Ministry of Science and Environmental Protection of the Republic of Serbia under Contract No. 1958 ("Transport processes of particles in fission and fusion systems").

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ЈЕДАН ЗАПИС О АНАЛИТИЧКИМ АПРОКСИМАЦИЈАМА Н-ФУНКЦИЈЕ

У раду су приказане неке раније изведене аналитичке апроксимације Н-функције добијене поступком раздвајања угаоне густине флуksа честица и применом DPN методе нултог реда, и представљено је једно ново решење засновано на модификованој DPN методи. Извршено је поређење са познатим Чандрасекаровим резултатима добијеним нумеричким поступком, као и са Помранинговим аналитичким апроксимацијама.