A METHOD OF APPROXIMATE GREEN'S FUNCTION FOR SOLVING REFLECTION OF PARTICLES IN PLANE GEOMETRY

by

Čedomir I. BELIĆ [1*]**,** ***Rodoljub D. SIMOVIĆ*** [2]**, and** ***Koviljka Dj. STANKOVIĆ*** [1]

[1] Faculty of Electrical Engineering, University of Belgrade, Belgrade, Serbia
[2] Vinča Institute of Nuclear Sciences, University of Belgrade, Belgrade, Serbia

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A method for approximate analytical solution of transport equation for particles in plane geometry is developed by solving Fredholm integral equations. Kernels of these equations are the Green's functions for infinite media treated approximately. Analytical approximation of Green's function is based on decomposition of the functions into terms that are exactly analytically solved and those which are approximately obtained by usual low order DPN approximation. Transport of particles in half-space is treated, and reflection coefficient is determined in the form of an analytical function. Comparison with the exact numerical solution and other approximate methods justified the proposed analytical technique.

Key words: transport equation, Green's function, DPN approximation, reflection coefficient

INTRODUCTION

Usual analytical treatment of the particle transport in a finite medium prefers application of Placzek lemma which enables formulation of integral equations for angular fluxes on boundary surfaces of different media [1, 2]. The aim of this work is to develop a method for analytical approximate determination of Green's function for infinite medium. It will enable to transform the application of Placzek lemma into solving the Fredholm integral equation or system of equations. In this way solution of singular integral equations or numerical solution of integral equations is omitted. That is way a method is proposed for decomposition of transport equation Green's function into terms that can be analytically exactly solved and terms which are determined by low order approximate solution of equation (DP0 and DP1 methods) [3]. Application of thus determined Green's function permits further analytical treatment of integral equations and the analytical solution of reflection coefficient for half-space. Proposed decomposition for solving transport equation is not bound to application of Placzek lemma, but it might be used for treatment of particle transport in finite media in the classical way with boundary conditions [4, 5].

The procedure is carried out with an assumption that the scattering in the medium is described by synthetic scattering law which allows isotropic particle scattering and scattering straight ahead [6-8]. For such a function it was possible to formulate similar procedure as for the case of strictly isotropic scattering of particles. Expressions for the Green's functions dependent on the space variable $x \geq 0$ are developed and solution of integral equation and determined reflection coefficient for half-space is presented. The obtained results are compared to the exact calculations and other verified approximate methods for solving the half-space transport problem [9, 10].

APPROXIMATE GREEN'S FUNCTION IN PLANE GEOMETRY

Let the particles scattering in a medium be described by the synthetic scattering law in the following form [7]

$$f(\mu,\mu') = \frac{1}{2\pi}\left[\frac{\ell}{2} + m\delta(\mu-\mu')\right] \qquad (1)$$

with conditions: $\ell + m = 1$ and $\ell, m \geq 0$. Here, μ and μ' are the cosines of particle angles befor and after scattering. Then the Green's function in plane geometry $G(x, \mu, x_0, \mu_0)$ is a solution of the transport equation

$$\mu\frac{\partial G(x,\mu,x_0,\mu_0)}{\partial x} + G(x,\mu,x_0,\mu_0) = \frac{c\ell}{2}\int_{-1}^{1} G(x,\mu',x_0,\mu_0)\mathrm{d}\mu' + cmG(x,\mu,x_0,\mu_0) + \delta(x-x_0)\delta(\mu-\mu_0) \qquad (2)$$

where c is the mean number of secondary particles per collision.

* Corresponding author; e-mail: belic.cedomir@gmail.com

Further, it will be considered that $x_0 = 0$.

From eq. (2) we separate the flux of unscattered particles G^0

$$G^{0}(x,\mu,\mu_0)\quad \frac{\mathrm{e}^{\ (1\quad cm)x/\mu}}{\mu}\quad \delta(\mu\quad \mu_0) \tag{3}$$

and split the solution of transport equation into two terms G^+ and G^- dependent on the particles directions. By applying Fourier transformation on eq. (2) we obtain

$$(1\quad \mathrm{c}m\quad \mathrm{i}k\mu)G\quad (k,\mu)$$
$$\frac{\mathrm{c}\ell}{2}\int_0^1 G\quad (k,\mu\ )\mathrm{d}\mu\quad \frac{\mathrm{c}\ell}{2}\int_1^0 G\quad (k,\mu\ )\mathrm{d}\mu$$
$$S\quad (k),\ \mu\quad 0, \tag{4}$$

and

$$S\quad (k)\quad \frac{\mathrm{c}\ell}{2}\frac{1}{1\quad \mathrm{c}m\quad \mathrm{i}k\mu_0} \tag{5}$$

$$G\quad (k,\mu)\quad \int_{\infty}^{\infty} G(x,\mu,x_0,\mu_0)\mathrm{e}^{\ ikx}\mathrm{d}x \tag{6}$$

where for the sake of brevity x_0 and μ_0 are omitted; i is the imaginary unit, and k – the complex variable.

In order to derive the Green's function approximately as an analytical function we will propose different shapes of G^+ and G^- decompositions, depending on the half-space $(x\quad 0)$ where these functions will be applied.

In the case of half-space $x > 0$ we use the following decomposition

$$G\quad (k,\mu)\quad \widetilde{G}\quad (k,\mu) \tag{7}$$

$$G\quad (k,\mu)\quad \sum_{l\ 1}^{L} G^{l}\quad (k,\mu)\quad \widetilde{G}\quad (k,\mu) \tag{8}$$

where G^{l-} presents the flux of l-times scattered particles moving in directions $\mu < 0$ after each collision. $\widetilde{G}$ and $\widetilde{G}$ are the components of Green's function calculated approximately by the method of double Legendre polynomials (DPN approximation)

$$\widetilde{G}\quad (x,\mu)\quad \sum_{n\ 0}^{N}(2n\quad 1)\widetilde{G}_n\ (x)P_n\ (2\mu\quad 1) \tag{9}$$

If the Green's functions should be applied in the $x < 0$ half-space, the following decompositions are used

$$G\quad (k,\mu)\quad \sum_{l\ 1}^{L} G^{l}\quad (k,\mu)\quad \widetilde{G}\quad (k,\mu) \tag{10}$$

$$G\quad (k,\mu)\quad \widetilde{G}\quad (k,\mu) \tag{11}$$

where G^{l+} represents the flux of l-times scattered particles mowing in directions $\mu > 0$ after each collision, while $\widetilde{G}$ and $\widetilde{G}$ are defined by (9).

The adventage of such decompositions of Green's functions is in the possibility of exact analytical determination of G^{l} and G^{l-} functions and the approximate analytical derivations of $\widetilde{G}$ and $\widetilde{G}$ components by low-order DPN method.

Green's function for $x > 0$

In Fourier transformed form components $G^{l-}(k,\mu)$ can be represented by

$$G^{l}\quad (k,\mu)\quad \frac{\mathrm{c}\ell}{2}\frac{a^{l\ \ 1}(k)}{(1\quad \mathrm{c}m\quad \mathrm{i}k\mu_0)(1\quad \mathrm{c}m\quad \mathrm{i}k\mu)} \tag{12}$$

where

$$a(k)\quad \frac{\mathrm{c}\ell}{2\mathrm{i}k}\ln(1\quad \mathrm{c}m\quad \mathrm{i}k) \tag{13}$$

Parts of Green's functions determined approximately by DP0 method have the following form

$$G_0\ (k)\quad \frac{\mathrm{c}\ell}{i\mu_0}\frac{A\quad (k)}{\Pi(k)} \tag{14}$$

where

$$A\quad (k)\quad [2\quad \mathrm{c}(2\mathrm{c}m\quad \ell)\quad \mathrm{i}k]\sum_{l\ 0}^{L\ 1} a^{l}(k)$$
$$(2\quad 2\mathrm{c}m\quad \mathrm{i}k)\,a^{L}(k) \tag{15}$$

$$A\quad (k)\quad \mathrm{c}\ell\sum_{l\ 0}^{L\ 1} a^{l}(k)\quad (2\quad 2\mathrm{c}m\quad \mathrm{i}k)\,a^{L}(k) \tag{16}$$

$$\Pi(k)\quad k\quad \frac{i}{\mu_0}(1\quad \mathrm{c}m)\quad k\quad \frac{i}{\nu_0}\quad k\quad \frac{i}{\nu_0} \tag{17}$$

$$\nu_0\quad \frac{1}{2}[1\quad \mathrm{c}m\quad \mathrm{c}(m\quad \ell)(1\quad m\ell)]^{\frac{1}{2}} \tag{18}$$

When inverse Fourier transformation is performed for $x > 0$ by contour integration in complex plane, the following Green's functions necessary for solving the transport problem of half-space are obtained

$$G(x,\mu,\mu_0)$$
$$\frac{\mathrm{c}\ell\nu_0\mu_0 A\quad [i(1\quad \mathrm{c}m)/\mu_0]}{2}\mathrm{e}^{\ (1\quad cm)x/\mu_0}$$
$$\frac{\mathrm{c}\ell\nu_0^2 A\quad (i/\nu_0)}{2[\mu_0\quad (1\quad \mathrm{c}m)\nu_0]}\mathrm{e}^{\ x/\nu_0},\ \mu_0\quad 0,\mu\quad 0 \tag{19}$$

$$G(x,\mu,\mu_0)$$
$$\frac{\mathrm{c}\ell}{2(1\quad \mathrm{c}m)(\mu\quad \mu_0)}\sum_{l\ 0}^{L\ 1} a^{l}[i(1\quad \mathrm{c}m)/\mu_0]\ \mathrm{e}^{\ (1\quad \mathrm{c}m)x/\mu_0}$$
$$\frac{\mathrm{c}\ell\nu_0^2\mu_0 A\quad [i(1\quad \mathrm{c}m)/\mu_0]}{\mu_0^2\quad [(1\quad \mathrm{c}m)\nu_0]^2}\mathrm{e}^{\ (1\quad cm)x/\mu_0}$$
$$\frac{\mathrm{c}\ell\nu_0^2 A\quad (i/\nu_0)}{2[\mu_0\quad (1\quad \mathrm{c}m)\nu_0]}\mathrm{e}^{\ x/\nu_0},\ \mu_0\quad 0,\mu\quad 0$$

(20)

$$G(x,\mu,\mu_0) \quad \frac{c\ell\nu_0^2 A \quad (i/\nu_0)}{2[\mu_0 \quad (1 \quad cm)\nu_0]} e^{\quad x/\nu_0}, \mu_0 \quad 0, \mu \quad 0 \quad (21)$$

$$G(x,\mu,\mu_0) \quad \frac{c\ell\nu_0^2 A \quad (i/\nu_0)}{2[\mu_0 \quad (1 \quad cm)\nu_0]} e^{\quad x/\nu_0}, \mu_0 \quad 0, \mu \quad 0 \quad (22)$$

Green's function for $x < 0$

Function G^{l+} can be written as

$$G^{l} \quad (k,\mu) \quad \frac{c\ell}{2} \frac{b^{l \quad 1}(k)}{(1 \quad cm \quad k\mu_0)(1 \quad cm \quad k\mu)} \quad (23)$$

where

$$b(k) \quad \frac{c\ell}{2ik} \ln(1 \quad cm \quad ik) \quad (24)$$

and the terms determined by DP0 method are

$$\widetilde{G}_O(k) \quad \frac{c\ell}{i\mu_0} \frac{B \quad (k)}{\Pi(k)} \quad (25)$$

where

$$B \quad (k) \quad c\ell \sum_{l \quad 0}^{L \quad 1} b^l(k) \quad (2 \quad 2cm \quad ik) b^L(k) \quad (26)$$

$$B \quad (k)$$

$$[2 \quad c(2cm \quad \ell) + ik] \sum_{l \quad 0}^{L \quad 1} b^l(k) \quad [2 \quad 2cm \quad ik] b^L(k) \quad (27)$$

For solving the reflection phenomena, we need Green's functions for $x < 0$, which are determined by inverse Fourier transformation

$$G(x,\mu,\mu_0) \quad \frac{c\ell\nu_0^2}{2} \frac{B \quad (\quad i/\nu_0)}{\mu_0 \quad (1 \quad cm)\nu_0} e^{x/\nu_0}, \mu_0 \quad 0, \mu \quad 0 \quad (28)$$

and

$$G(x,\mu,\mu_0) \quad \frac{c\ell\nu_0^2}{2} \frac{B \quad (\quad i/\nu_0)}{\mu_0 \quad (1 \quad cm)\nu_0} e^{x/\nu_0}, \mu_0 \quad 0, \mu \quad 0 \quad (29)$$

Numerical results for a plane monodirectional source

In tab. 1 integral values of Green's functions are shown for $x > 0$, $c = 0{,}5$, $\mu_0 = 1$, and isotropic scattering law ($\ell \quad 1, m \quad 0$). They are obtained by the developed method of approximate Green's function and compared to the four digit exact values [9]. Method of the approximate Green's function is applied in case of $L \quad \infty$ with the double Legendre approximations obtained by DP0 and DP1 methods. It is evident that in the vicinity of the source the approximate Green's function method gives very precise values. For $x = 0$, in combination with the DP0 method, deviation from the exact value is about 0.5 %, while in combination with the DP1 method the accuracy is to the unit of the fourth digit (0.017 %). In the latter case (with DP1 method), the accuracy is very satisfactory even at great distances from the source.

ALBEDO FOR A HALF-INFINITE SPACE

Determination of the reflection coefficient

We supose that the medium fills up the half-space $x > 0$, that the incoming flux is $\phi^+(\mu) = \delta(\mu - \mu_0)$, the outgoing flux is $\phi^-(\mu)$, and that there are no active sources in the medium. Application of the Placzek lemma leads to the integral equation [1]

$$\phi \quad (\mu) \quad \int_0^1 \phi \quad (\mu') G(0,\mu,\mu')\mu' d\mu'$$

$$\int_1^0 \phi \quad (\mu') G(0,\mu,\mu')\mu' d\mu', \quad \mu \quad 0 \quad (30)$$

where Green's functions are obtained from expressions (20) and (22) for $x = 0$. Owing to the simple analytical forms of approximate Green's functions, eq. (30) represents the Fredholm integral equation of the second kind that can be easily solved. By defining the reflection coefficient in usual way as the ratio between the outgoing and the incoming currents

$$R \quad \frac{\int_1^0 \phi \quad (\mu')\mu \, d\mu}{\int_0^1 \phi \quad (\mu')\mu \, d\mu} \quad (31)$$

Table 1. Comparison of exact and approximate Green's function for isotropicaly scattered particles and a plane parallel monodirectional source ($c = 0{,}5$, $\mu_0 = 1$)

x*	$G^l \quad \widetilde{G}_{DP0}$	$G^l \quad \widetilde{G}_{DP1}$	Exact [9]
0	0.2980	0.2995	0.29955
0.25	0.3643	0.3890	0.3870
0.50	0.3766	0.3872	0.3813
1.0	0.3250	0.3113	0.3105
3.0	$0.7597 \; 10^{-1}$	$0.7584 \cdot 10^{-1}$	$0.7590 \cdot 10^{-1}$
5.0	$0.1217 \cdot 10^{-1}$	$0.1450 \cdot 10^{-1}$	$0.1437 \cdot 10^{-1}$
10.0	$0.9064 \cdot 10^{-4}$	$0.1617 \cdot 10^{-3}$	$0.1673 \cdot 10^{-3}$

* x is measured in mean free path

we obtain reflection coefficient R for isotropic scattering medium

$$R(\mu_0) \quad \frac{1}{2}[A(\mu_0,\nu_0) \quad B(\mu_0)\, I_1(\mu_0)]$$
$$\frac{1}{2}\frac{C(\nu_0)}{1 \quad C(\nu_0) I_1(\nu_0)}$$
$$A(\mu_0,\nu_0)\, I_1(\nu_0) \quad B(\mu_0)\frac{I_1(\nu_0) \quad I_1(\mu_0)}{\nu_0 \quad \mu_0} \tag{32}$$

where new functions are defined as

$$I_1(x) \quad 1 \quad x \ln \quad 1 \quad \frac{1}{x} \tag{33}$$

$$A(x,y) \quad \frac{c^2 y^2}{x \quad y} \quad \frac{x}{(x \quad y) I_2(x)} \quad \frac{1}{2 I_2(y)} \tag{34}$$

$$B(x) \quad \frac{x}{2 I_2(x)} \tag{35}$$

$$C(x) \quad \frac{c^2 x^2}{2 I_2(x)} \tag{36}$$

$$I_2(x) \quad 1 \quad \frac{cx}{2} \ln \quad 1 \quad \frac{1}{x} \tag{37}$$

Numerical results for various values of c and μ_0

In tab. 2 the reflection coefficients obtained by the DP0 and DP1 methods [10] and our results for $L \quad \infty$, are compared to the four digits exact values [10]. It is obvious that the approximate Green's function method is superior to DP0 approximation. This is true for DP1 approximation only for high absorption media (c 0,3). For high scattering media (c 0,9) approximate Green's function method gives less acurate values of reflection coefficient but the relative error is always less than 1 %.

CONCLUSION

Here developed method enables analytical treatment of the transport processes in plane geometry. The results obtained for angular integral of Green's function in infinite medium and reflection coefficients of half-space show the efficiency of the approximate Green's function method in different domains of parametres c and μ_0. It is shown that the less precise results obtained for angular integral of Green's function for greater distances from the source could be improved by applying higher order of DPN approximation (*e. g.*, DP1 instead of DP0).

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AUTHORS' CONTRIBUTIONS

Theoretical analysis carried out and manuscript was written by Č. I. Belić. All authors analyzed and discussed the results.

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Table 2. Comparison of the reflection coefficients of a half-space obtained by the DP0 and DP1 methods, our approximation, and the exact method

c	μ_0	De Corte and Mertens [10]		This work	Exact [10]
		DP0	DP1		
0.1	0.0	0.05132	0.05132	0.05132	0.05132
	0.5	0.02633	0.02406	0.02387	0.02388
	1.0	0.01771	0.01642	0.01636	0.01639
0.3	0.0	0.16334	0.16334	0.13664	0.16334
	0.5	0.08893	0.08217	0.08160	0.08172
	1.0	0.06110	0.05726	0.05694	0.05721
0.5	0.0	0.29289	0.29289	0.29299	0.29289
	0.5	0.17157	0.16072	0.15980	0.16013
	1.0	0.12132	0.11518	0.11434	0.11521
0.7	0.0	0.45228	0.45228	0.45289	0.45228
	0.5	0.29222	0.27862	0.27759	0.27812
	1.0	0.21584	0.20842	0.20663	0.20867
0.9	0.0	0.68377	0.68377	0.68718	0.68377
	0.5	0.51949	0.50807	0.50877	0.50795
	1.0	0.41886	0.41438	0.41183	0.41495

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Чедомир И. БЕЛИЋ, Родољуб Д. СИМОВИЋ, Ковиљка Ђ. СТАНКОВИЋ

МЕТОДА АПРОКСИМАТИВНЕ ГРИНОВЕ ФУНКЦИЈЕ ЗА РЕШАВАЊЕ РЕФЛЕКСИЈЕ ЧЕСТИЦА У РАВНОЈ ГЕОМЕТРИЈИ

Метода апроксимативног аналитичког решења транспортне једначине честица у равној геометрији развијена је решавањем Фредхолмових интегралних једначина. Језгра ових једначина су Гринове функције бесконачне средине апроксимативно одређене. Аналитичка апроксимација Гринових функција заснива се на разлагању функција на чланове који су егзактно аналитички решени и оне који су апроксимативно одређени уобичајеном ДПН апроксимацијом ниског реда. Разматран је транспорт честица у полупростору и одређен је коефицијент рефлексије у виду аналитичке функције. Поређење са егзактним нумеричким решењем и другим апроксимативним методама оправдава овде развијен аналитички поступак.

Кључне речи: транспортна једначина, Гринова функција, ДПН апроксимација, коефицијент рефлексије